

SLE($\kappa, \vec{\rho}$) and Conformal Field Theory

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Abstract

SLE($\kappa, \vec{\rho}$) is a generalisation of Schramm-Loewner evolution which describes planar curves which are statistically self-similar but not conformally invariant in the strict sense. We show that, in the context of boundary conformal field theory, this process arises naturally in models which contain a conserved U(1) current density J^μ , in which case it gives rise to a highest weight state $|h\rangle$ satisfying a deformation $2L_{-2}|h\rangle = (\kappa/2)L_{-1}^2|h\rangle + \alpha J_{-1}L_{-1}|h\rangle$ of the usual level 2 null state condition.

We apply this to a free field theory with piecewise constant Dirichlet boundary conditions, with a discontinuity λ at the origin, and argue that this will lead to level lines in the bulk described by SLE($4, \vec{\rho}$) across which there is a universal macroscopic jump $\pm\lambda^*$ in the field, independent of the value of λ .

1 Introduction

In recent years, Schramm-Loewner evolution (SLE)[1, 2, 3, 4] has revolutionised the study of the continuum limit of two-dimensional critical systems. This approach focuses on finding the correct probability measure to describe the random curves, such as cluster boundaries, which occur in such systems. In the simplest setting of curves which connect two distinct points z_1, z_2 on the boundary of a simply connected domain \mathcal{D} , this measure is generated dynamically by evolving the curve starting from one end point. Conventionally the domain is taken to be the upper half plane \mathbf{H} , and the curve γ_t as evolved up to time t (or rather its hull K_t , which includes any regions enclosed by the curve) is characterised by the conformal mapping $g_t : \mathbf{H} \setminus K_t \rightarrow \mathbf{H}$, which is unique if we demand that $g_t(z) \sim z + 2t/z + O(z^{-2})$ at infinity. This function satisfies the Loewner equation

$$\frac{dg_t(z)}{dt} = \frac{2}{g_t(z) - W_t},$$

where the continuous function W_t is the image of the growing tip of the curve under g_t . In this paper we shall find it more useful to define $\hat{g}_t(z) \equiv g_t(z) - W_t$, which always maps the growing tip of the curve to the origin, and satisfies $d\hat{g}_t = 2dt/\hat{g}_t - dW_t$.

This is completely general. However, Schramm[1] argued that if the following conditions hold:

- A. If \mathcal{D} is mapped conformally to another domain \mathcal{D}' , so that a curve γ is mapped to γ' , then the induced measure on γ' is the correct measure for γ' in \mathcal{D}' ;
- B. If γ_t is the part of the curve up time t , then the conditional measure on the remainder $\gamma \setminus \gamma_t$ in \mathcal{D} is the same as the unconditional measure in $\mathcal{D} \setminus K_t$;

together with a reflection property, then the only possibility for the driving term W_t is Brownian motion, namely $dW_t = \sqrt{\kappa}dB_t$, where B_t is standard Brownian motion and κ is the diffusion constant. Different values of κ correspond to different universality classes of critical behaviour.

The continuum limit of isotropic short-range 2d critical systems is also believed to be described by conformal field theory (CFT). This focuses on the correlation functions of local operators which are the scaling limits of local lattice observables. Within radial quantisation and the operator-state correspondence of CFT, these correspond to states in the Hilbert space of the theory which fall into highest weight representations of the Virasoro algebra. This approach appears to be very different from that of SLE. Nevertheless it can be argued (see later) that to each partial curve γ_t can be associated a state in this space. The ensemble of curves satisfying conditions (A) and (B) then corresponds to a highest weight state $|h\rangle$ whose Verma module contains a level 2 null state: $(L_{-2} - (\kappa/4)L_{-1}^2)|h\rangle = 0$. This implies, using the BPZ[5] equations, that expectation values of all observables which can be written as $\langle \mathcal{O} | h \rangle$ satisfy linear second-order differential equations. These are the same equations found from the stochastic approach. This partial unification between the two approaches has been very fruitful, both in understanding CFT and in suggesting possible generalisations of SLE to other domains[6] and to multiple curves[7].

One of the important properties of Brownian motion is scale invariance: $\sigma^{-1}W_{\sigma^2 t}$ has the same law as W_t . This extends to the whole sequence of Loewner mappings: $\sigma^{-1}g_{\sigma^2 t}(\sigma z)$ obeys the same equation as g_t . This means that the curves described by this process are statistically self-similar (which of course is a special case of conformal symmetry.) $\text{SLE}(\kappa, \vec{\rho})$ is a minimal way of generalising SLE while retaining self-similarity. This is done by extending the stochastic equation satisfied by W_t to the system

$$dW_t = \sqrt{\kappa} dB_t - \sum_{j=1}^n \frac{\rho_j dt}{X_t^{(j)}}; \quad (1)$$

$$dX_t^{(j)} = \frac{2dt}{X_t^{(j)}} - dW_t, \quad (2)$$

where the parameters $\vec{\rho} \equiv (\rho_1, \dots, \rho_n)$ are constants. The auxiliary variables $X_t^{(j)} = \hat{g}_t(X_0^{(j)})$ are not independent of W_t : indeed they could be integrated out. The curve γ will thus depend not only on the particular realisation of the Brownian motion but also the initial values $Z_0^{(j)}$ as well as the $\{\rho_j\}$.

$\text{SLE}(\kappa, \vec{\rho})$ processes were first introduced in Ref. [8] as examples of restriction measures. Their properties have been studied further in [9, 10], and they were used in [11] in order to study certain properties of SLE which also lead to a

proof of Watts' formula[12] for percolation.

Let us first remark that the equations (1,2) are just special cases of a more compactly stated problem. Let $\Phi_t(z)$ be the (unique up to an additive constant) harmonic function in the upper half plane which, on the real axis, is piecewise constant with discontinuities $\pi\rho_j$ at the points $X_t^{(j)}$. Define the current density $\mathcal{J}_t^\mu \equiv \epsilon^{\mu\nu}\partial_\nu\Phi_t$, which is conserved everywhere except at the sources $X_t^{(j)}$. Then (1) may be written

$$dW_t = \sqrt{\kappa}dB_t - \mathcal{J}_t^x(0)dt. \quad (3)$$

Note that while Φ_t is conformally invariant, in the sense that $\Phi_t(\hat{g}_t^{-1}(z))$ is harmonic in $\mathbf{H} \setminus K_t$, the measure on curves given by (3) is *not* invariant, because the derivative in $\mathcal{J}_t^x = \partial_y\Phi_t$ is with respect to z , not $\hat{g}_t^{-1}(z)$.

We shall use (3) as a general definition of SLE(κ, ρ).¹ One advantage is that it may be used to define this process in other domains \mathcal{D} , where \hat{g}_t maps $\mathcal{D} \setminus K_t$ onto \mathcal{D} , and the current density \mathcal{J}^μ is evaluated in this domain. Moreover, as will become apparent, there are situations when (4) applies even though there are no sources for \mathcal{J}^μ .

In this paper we extend the correspondence between random curves and CFT to include those described by SLE(κ, ρ), as defined by (3). It turns out that the null state condition is replaced by a more general one

$$(2L_{-2} - (\kappa/2)L_{-1}^2 - \mathcal{J}_{-1}L_{-1})|h\rangle = 0, \quad (4)$$

where $\mathcal{J}_{-1} = \mathcal{J}_0^x(0)$. This is described in Sec. 2.

These equations are more likely to be physically relevant if the current density \mathcal{J}^μ arises naturally in the theory under consideration. A simple example is the theory of a free gaussian field ϕ , with action $S = (g/4\pi) \int (\partial\phi)^2 d^2z$ and with piecewise constant Dirichlet boundary conditions. This is described in Sec. 3. The jumps in the boundary condition act as sources for the current density $J^\mu \propto \epsilon^{\mu\nu}\partial_\nu\phi$. In particular we consider the effect of introducing a jump of strength λ at the origin, corresponding to the insertion of a boundary condition changing operator $\phi_\lambda(0)$. We show that there is a special value $\lambda^* = (4g)^{-1/2}$ for which the corresponding highest weight state satisfies the usual level 2 null state condition with $\kappa = 4$. Its conformal weight $h = g\lambda^{*2}$

¹In this formulation the parameters $\vec{\rho}$ are hidden. However in this paper the original name will be used even when the sources are not specified.

takes the value $\frac{1}{4}$. For other values of λ , however, it satisfies the deformed condition (4), with $\mathcal{J} = \alpha J$ where $\alpha = ((\lambda^*/\lambda) - (\lambda/\lambda^*))$, and J normalised so that a discontinuity λ^* corresponds to unit charge.

Of course, in this theory, changing λ is equivalent to changing the coupling constant g , which is the same as perturbing the action with the exactly marginal operator $(\partial\phi)^2 \propto J\bar{J}$. From this point of view, the perturbation partially screens the current-current correlations. We show that this screening acts to renormalise the effective U(1) charge λ to the critical value λ^* , with effective conformal weight $\frac{1}{4}$.

In Sec. 3.2 we consider the physical implications of this. The free field theory should correspond to the continuum limit of a gaussian free field on a lattice, for which it is possible to identify uniquely the level lines of the field. For this theory we argue that the deformed condition (4) is satisfied by the state corresponding to curves which are the level lines of the free field emanating from the discontinuity λ in ϕ at the origin. If $\lambda = \pm\lambda^*$ it does not matter which level line, because there is a macroscopic jump $\pm\lambda^*$ in the field ϕ across the curve in the bulk. This curve should be described by SLE₄, as proved by Sheffield and Schramm[13].

We shall argue that if $\lambda \neq \lambda^*$ then the level lines are described by SLE($\kappa, \vec{\rho}$). However, the screening alluded to above has the effect of renormalising the effective jump across the curve in the bulk to the universal value λ^* , rather than λ . The physics of this is clearer in the discrete gaussian model, where the values of $\phi(r)$ at lattice sites are integer multiples of some unit $2\pi\Lambda$.

2 Random curves and CFT

We start by discussing how the measure on curves γ can be used generate a highest weight state in boundary CFT. Our point of view is slightly different from that of Ref. [14], being better suited to the generalisation to SLE($\kappa, \vec{\rho}$). In BCFT, we suppose that there is some set of fundamental local fields $\psi(r)$ (the continuum limit of the local lattice degrees of freedom), satisfying given conformally invariant boundary conditions on the real axis (with the possible exception of the origin) and with a unnormalised Gibbs measure $e^{-S[\psi]}[d\psi]$. The Hilbert space is that of all possible field configurations ψ_Γ on a fixed semicircle Γ centred on the origin. The vacuum state is given by weighting

each state $|\psi'_\Gamma\rangle$ by the (normalised) path integral restricted to the interior of Γ and conditioned on the fields taking the specified values ψ'_Γ on the boundary:

$$|0\rangle = \int [d\psi'_\Gamma] \int_{\psi_\Gamma=\psi'_\Gamma} [d\psi] e^{-S[\psi]} |\psi'_\Gamma\rangle.$$

Similarly, inserting a local operator $\phi(0)$ at the origin into the path integral defines a state $|\phi\rangle$. This is the well-known operator-state correspondence of CFT. Because of scale invariance, a different choice for Γ gives the same states, up to a multiplicative constant.

Now suppose that for every field configuration ψ we can identify a curve γ connecting the origin to infinity. The existence of such a curve is assumed to be guaranteed by the boundary conditions on the real axis. On the lattice, for example if γ is a cluster boundary in the Ising model, we take the spins on the negative real axis to be -1 , and those on the positive real axis to be $+1$. We assume this property continues to hold in the continuum limit. Any such curve may be generated by a Loewner process: denote as before the part of the curve up to time t by γ_t . The existence of this curve depends on only the field configurations ψ in the interior of Γ , as long as γ_t lies wholly inside this region. Then we can condition the fields contributing to the path integral on the existence of γ_t , thus defining a state

$$|\gamma_t\rangle = \int [d\psi'_\Gamma] \int_{\psi_\Gamma=\psi'_\Gamma; \gamma_t} [d\psi] e^{-S[\psi]} |\psi'_\Gamma\rangle.$$

The path integral (over the whole of the upper half plane, not just the interior of Γ), when conditioned on γ_t , gives a measure $d\mu(\gamma_t)$. The state

$$|h\rangle = |h_t\rangle \equiv \int d\mu(\gamma_t) |\gamma_t\rangle$$

is in fact independent of t , since it is just given by the path integral conditioned on there being a curve connecting the origin to infinity, which is guaranteed by the boundary conditions. In fact, by taking $t = 0$, we see that $|h\rangle$ is just the state corresponding to a boundary condition changing operator[15] at the origin.

However, $d\mu(\gamma_t)$ is also given by the measure on W_t in Loewner evolution, through the iterated sequence of conformal mappings satisfying $d\hat{g}_t = 2dt/\hat{g}_t - dW_t$. This corresponds to an infinitesimal conformal mapping of

the upper half plane minus γ_t . In CFT, this is implemented by considering a more general transformation $x^\mu \rightarrow x^\mu + \alpha^\mu(x)$, where $\alpha^\mu(x)$ agrees with the conformal transformation inside a contour C which surrounds K_t but lies inside Γ , but vanishes outside C . This is compensated by inserting $(1/2\pi i) \int_C \alpha(z) T(z) dz + \text{c.c.}$ into the path integral, where $T(z)$ is the local stress tensor. Because of the conformal boundary condition that $T = \bar{T}$ on the real axis, we may drop the second c.c. term by extending C to include its reflection in the real axis. In our case, $d\hat{g}_t$ corresponds to inserting $(1/2\pi i) \int_C (2dt/z - dW_t) T(z) dz$. In operator language, this corresponds to acting on $|\gamma_t\rangle$ with $2L_{-2}dt - L_{-1}dW_t$ where $L_n = (1/2\pi i) \int_C z^{n+1} T(z) dz$. Thus, for any $t_1 < t$,

$$|g_{t_1}(\gamma_t)\rangle = \mathbf{T} \exp \left(\int_0^{t_1} (2L_{-2}dt' - L_{-1}dW_{t'}) \right) |\gamma_t\rangle,$$

where \mathbf{T} denotes a time-ordered exponential.

The measure on γ_t is the product of the measure of $\gamma_t \setminus \gamma_{t_1}$, conditioned on γ_{t_1} , with the unconditioned measure on γ_{t_1} . The first is the same as the unconditioned measure on $g_{t_1}(\gamma_t)$, and the second is given by the measure on $W_{t'}$ for $t' \in [0, t_1]$. Thus

$$|h_t\rangle = \int d\mu(g_{t_1}(\gamma_t)) \int d\mu(W_{t'}; t' \in [0, t_1]) \mathbf{T} e^{-\int_0^{t_1} (2L_{-2}dt' - L_{-1}dW_{t'})} |g_{t_1}(\gamma_t)\rangle.$$

For ordinary SLE, W_t is proportional to a Brownian process. The integration over realisations of this for $t' \in [0, t_1]$ may be performed by breaking up the time interval into small segments, expanding out the exponential, using $(dB_{t'})^2 = dt'$, and re-exponentiating. The result is

$$|h_t\rangle = \exp \left(-(2L_{-2} - (\kappa/2)L_{-1}^2)t_1 \right) |h_{t-t_1}\rangle.$$

But, as we argued from the path integral, $|h_t\rangle$ is independent of t , and therefore

$$(2L_{-2} - (\kappa/2)L_{-1}^2)|h\rangle = 0, \tag{5}$$

that is, there is a level 2 null state. Note that $|h\rangle = |h_0\rangle$ is also expected to be a highest weight state, $L_n|h\rangle = 0$ ($n \geq 1$), since in this case we can shrink C to zero.

For $\text{SLE}(\kappa, \rho)$ the measure depends on the values of $\{X_0^{(j)}\}$ and therefore so does the state $|h_t; \{X_0^{(j)}\}\rangle$. In writing how this state behaves under the

infinitesimal conformal mapping $d\hat{g}_t$, we need to decide whether the points $\{X_t^{(j)}\}$ lie inside or outside the contour C . In order to be able to define a highest weight state, we need to be able to shrink C to the origin without obstructions, so it is natural to choose it to lie inside all the $\{X_t^{(j)}\}$. But this has the (convenient) consequence that these points do not evolve under the modified transformation which vanishes outside C . Thus the drift term in (1) is constant, and it is still straightforward to integrate over the measure on $dB_{t'}$, to obtain

$$|h_t; \{X_0^{(j)}\}\rangle = \exp\left(- (2L_{-2} - (\kappa/2)L_{-1}^2 + \sum_j (\rho_j/X_0^{(j)})L_{-1})t_1\right) |h_{t-t_1}; \{X_0^{(j)}\}\rangle,$$

so we can argue, as before, that there is a time-independent state $|h; \{X_0^{(j)}\}\rangle$ satisfying

$$\left(2L_{-2} - (\kappa/2)L_{-1}^2 + \sum_j (\rho_j/X_0^{(j)})L_{-1}\right) |h; \{X_0^{(j)}\}\rangle = 0. \quad (6)$$

Note that if we had taken C to lie outside all the $X_t^{(j)}$ this would incur the replacement

$$\begin{aligned} L_{-2} &\rightarrow L_{-2} + \sum_j (2/X_{t'}^{(j)}) \partial_{X_{t'}^{(j)}}, \\ L_{-1} &\rightarrow L_{-1} + \sum_j \partial_{X_{t'}^{(j)}}, \end{aligned}$$

corresponding to diffusion in the moduli space of the half-plane with marked boundary points, as well as the usual diffusion[6]. In addition, the evolved state would not be of highest weight. There is no contradiction here: the points $\{X_t^{(j)}\}$ act as the location of boundary condition changing operators $\Phi_j(X_t^{(j)})$, and the state we get by taking C to lie outside these points is not of highest weight and transforms non-trivially.

The last term in (6) may be written in terms of the current density $\mathcal{J}(z) = \sum_j \rho_j/(z - X_0^{(j)})$ introduced in Sec. 1. Its value at the origin is $\mathcal{J}_{-1} = (1/2\pi i) \int_C z^{-1} J(z) dz$. This gives (4).

3 Free field theory

In this section we illustrate the above for the simplest case of a free field theory. Consider field $\phi(r)$ in the upper half plane with action $S[\phi] = (g/4\pi) \int (\partial\phi)^2 d^2z$. The boundary conditions are piecewise Dirichlet: however there are discontinuities with jumps $2\pi\lambda_j$ at points x_j with $j = (0, 1, \dots, n)$:

$$\phi(x) = 2\pi \sum_j \lambda_j H(x - x_j),$$

where $H(x) = 1$ for $x > 0$ and 0 for $x < 0$. These boundary conditions are satisfied by the harmonic function

$$\phi_c(z, \bar{z}) = -2 \sum_j \lambda_j \arg(z - x_j) = i \sum_j \lambda_j \ln((z - x_j)/(\bar{z} - x_j)),$$

and if we write $\phi = \phi_c + \phi'$, with $\phi' = 0$ on the boundary, $S[\phi] = S[\phi_c] + S[\phi']$, so that the partition function is $Z = Z_\lambda Z'$, where

$$Z_\lambda = \prod_{j < k} ((x_k - x_j)/a)^{2g\lambda_j\lambda_k}, \quad (7)$$

where a is the UV cutoff, and Z' is the partition function for homogeneous Dirichlet boundary conditions.

In BCFT[15], we can think of Z_λ/Z' as the correlation function $\langle \prod_j \phi_{\lambda_j}(x_j) \rangle$ of boundary condition changing operators. Thus any expectation value $\langle \mathcal{O} \rangle$ of some observable with the inhomogeneous boundary conditions can be written

$$\langle \mathcal{O} \rangle_\lambda = \frac{\langle \mathcal{O} \prod_j \phi_{\lambda_j}(x_j) \rangle}{\langle \prod_j \phi_{\lambda_j}(x_j) \rangle}.$$

In particular, we can compute the expectation value of the stress tensor

$$\langle T(z) \rangle_\lambda = -g(\partial_z \phi_c)^2 = g \sum_j \sum_k \frac{\lambda_j \lambda_k}{(z - x_j)(z - x_k)},$$

and compare its behaviour as $z \rightarrow x_i$ with that expected from the conformal Ward identity[5]

$$T(z) \phi_{\lambda_i}(x_i) = \frac{h_i}{(z - x_i)^2} \phi_{\lambda_i}(x_i) + \frac{1}{z - x_i} \partial_{x_i} \phi_{\lambda_i}(x_i) + L_{-2} \phi_{\lambda_i}(x_i) + O(z - x_i).$$

An explicit computation gives

$$\begin{aligned} \langle T \rangle_\lambda = & \frac{g\lambda_i^2}{(z-x_i)^2} + \frac{2g}{z-x_i} \sum_j' \frac{\lambda_i \lambda_j}{x_i - x_j} \\ & - 2g \sum_j' \frac{\lambda_i \lambda_j}{(x_i - x_j)^2} + g \sum_{j,k}' \frac{\lambda_j \lambda_k}{(x_i - x_j)(x_i - x_k)} + O(z - x_i) \end{aligned}$$

where a prime on the sum omits the terms with $j, k = i$. From this we identify the scaling dimension of ϕ_{λ_i} to be

$$h_i = g\lambda_i^2,$$

and see that

$$\partial_{x_i} \ln \langle \prod_j \phi_{\lambda_j}(x_j) \rangle = 2g \sum_j' \frac{\lambda_i \lambda_j}{x_i - x_j},$$

which is of course consistent with (7).

We also see that

$$\langle L_{-2} \phi_{\lambda_i}(x_i) \prod_j' \phi_{\lambda_j}(x_j) \rangle = -2g \sum_j' \frac{\lambda_i \lambda_j}{(x_i - x_j)^2} + g \sum_{j,k}' \frac{\lambda_j \lambda_k}{(x_i - x_j)(x_i - x_k)},$$

while

$$\begin{aligned} \langle L_{-1}^2 \phi_{\lambda_i}(x_0) \prod_j' \phi_{\lambda_j}(x_j) \rangle &= \prod_j' (x_i - x_j)^{-2g\lambda_i \lambda_j} \partial_{x_i}^2 \prod_j' (x_i - x_j)^{2g\lambda_i \lambda_j} \\ &= -2g \sum_j' \frac{\lambda_i \lambda_j}{(x_i - x_j)^2} + 4g^2 \sum_{j,k}' \frac{\lambda_i^2 \lambda_j \lambda_k}{(x_i - x_j)(x_i - x_k)}. \end{aligned}$$

The condition $2L_{-2}\phi_{\lambda_i} = (\kappa/2)L_{-1}^2\phi_{\lambda_i}$ is satisfied (as an operator condition, that is for all choices of the λ_j and x_j), only if

$$\kappa = 4 \quad \text{and} \quad \lambda_i^2 = \lambda^{*2} = 1/4g,$$

so that $h_i = \frac{1}{4}$.

If the latter condition is not satisfied we have instead

$$2L_{-2}\phi_{\lambda_i} = 2L_{-1}^2\phi_{\lambda_i} - \sum_j' \frac{\rho_j}{x_j - x_i} L_{-1}\phi_{\lambda_i}, \quad (8)$$

where

$$\rho_j = (\lambda_j/\lambda_i) \left(1 - (\lambda_i/\lambda^*)^2\right) .$$

This has the form of (6), realised on operators rather than states, with $x_i = 0$, $X_0^{(j)} = x_j - x_i$. It is equally unsatisfactory as a local condition on ϕ_{λ_i} , but can be written, as (4), in terms of \mathcal{J}_{-1} . However in this theory, this is, up to a multiplicative factor, the conserved U(1) current density $J^\mu \propto \epsilon^{\mu\nu} \partial_\nu \phi$, for which the discontinuities at the boundaries act as local sources. It is useful to normalise this current so that

$$J(z)\phi_{\lambda^*}(0) \sim (1/z)\phi_{\lambda^*}(0)$$

(where $J = J_z$), so that a jump of λ^* has unit U(1) charge. This means taking $J = -2ig^{1/2}\partial_z\phi$.

As for $T(z)$, we can define the modes $J_n = (1/2\pi i) \int_C z^n J(z) dz$. The term in $\partial_{x_i}\phi_c$ with $j = i$ corresponds to J_0 : the rest is J_{-1} . (8) may then be rewritten as an operator condition on a highest weight state $|h_\lambda\rangle$:

$$(2L_{-2} - 2L_{-1}^2 - \alpha J_{-1}L_{-1})|h_\lambda\rangle, \quad (9)$$

where $\alpha = q_\lambda^{-1} - q_\lambda$ with $q_\lambda = \lambda/\lambda^*$.

Although (9) has been derived for the specific case when the current J is produced by sources which are themselves boundary condition changing operators, it is valid completely generally as an operator condition. With the normalisation of the current chosen above we have $T = -g :(\partial_z\phi)^2 := \frac{1}{4} :J^2 :$ where $J \equiv J_z$. Thus, in terms of operators,

$$L_n = \frac{1}{4} \sum_r :J_r J_{n-r} :,$$

where now the normal ordering symbol $:J_k J_l :$ places J_k to the right of J_l if $k > l$. Let $|h, q\rangle$ be a highest weight state of conformal weight h and U(1) charge q , so that $L_0|h, q\rangle = h|h, q\rangle$, $J_0|h, q\rangle = q|h, q\rangle$, and $L_n|h, q\rangle = J_n|h, q\rangle = 0$ for $n > 0$. Then

$$\begin{aligned} L_{-2}|h, q\rangle &= \frac{1}{4}(2J_{-2}J_0 + J_{-1}^2)|h, q\rangle = \frac{1}{4}(2qJ_{-2} + J_{-1}^2)|h, q\rangle \\ L_{-1}|h, q\rangle &= \frac{1}{4}(2J_{-1}J_0 + 2J_{-2}J_1)|h, q\rangle = \frac{1}{4}(2qJ_{-1})|h, q\rangle \\ L_{-1}^2|h, q\rangle &= \frac{1}{16}(2J_{-1}J_0 + 2J_{-2}J_1)(2qJ_{-1})|h, q\rangle = \frac{1}{4}(q^2J_{-1}^2 + kqJ_{-2})|h, q\rangle \\ J_{-1}L_{-1}|h, q\rangle &= \frac{1}{2}qJ_{-1}^2|h, q\rangle, \end{aligned}$$

where we have introduced the U(1) anomaly k by $\langle J(z)J(0) \rangle = k/z^2$, so that $[J_n, J_m] = kn\delta_{n,-m}$. An explicit calculation gives $k = 2$ with this normalisation for J , so that on forming the difference $(2L_{-2} - 2L_{-1})|h, q\rangle$, we again obtain (9), but completely generally.

Note that in the whole of this section, instead of considering Dirichlet boundary conditions with boundary condition changing operators ϕ_λ , we could, in the dual description, have considered Neuman boundary conditions on the dual field $\tilde{\phi}$, with insertions of vertex operators $e^{i\lambda\tilde{\phi}}$.

3.1 $J\bar{J}$ perturbation

Let us make the simple observation that if $\lambda \neq \lambda^*(g) = 1/2g^{1/2}$, we can always make it so by suitably changing g . Pick some reference value g_0 , so that action is $S_0 = (g_0/4\pi) \int (\partial\phi)^2 d^2z$. With respect to this action, a jump of $\lambda^* = 1/2g_0^{1/2}$ is a unit source for the current density with components $J = -2ig_0^{1/2}\partial_z\phi$, $\bar{J} = 2ig_0^{1/2}\partial_{\bar{z}}\phi$. Now consider the perturbed action

$$S = S_0 + u \int J\bar{J}d^2z = (g_{\text{eff}}(u)/4\pi) \int (\partial\phi)^2 d^2z,$$

where $g_{\text{eff}}(u) = g_0(1 + 4\pi u)$.

In the perturbed theory, the scaling dimension of ϕ_{λ^*} is modified to $g_{\text{eff}}\lambda^{*2} = \frac{1}{4}(1 + 4\pi u)$. The jumps corresponding to $h = \frac{1}{4}$ are now $\pm\lambda^*(1 + 4\pi u)^{-1/2}$.

Furthermore, the current-current correlations are partially screened. In the unperturbed theory

$$\langle J(z)J(0) \rangle = k/z^2 \quad \text{and} \quad \langle \bar{J}(\bar{z})\bar{J}(0) \rangle = k/\bar{z}^2,$$

where $k = 2$. This implies, for example, that

$$\langle J_y(x, y)J_y(0, 0) \rangle = -\frac{x^2 - y^2}{(x^2 + y^2)^2},$$

so that two currents alongside each other are more likely to be anti-parallel rather than parallel. This is because there are many small closed current loops. In the perturbed theory, keeping the normalisation of the current the same, the U(1) anomaly is reduced (if $u > 0$) by a factor $(1 + 4\pi u)^{-1}$. This is as if the effective current along a given loop were reduced by $(1 + 4\pi u)^{-1/2}$.

This means that the effective charge of ϕ_{λ^*} corresponds to a scaling dimension $\frac{1}{4}$.

In the perturbed theory we now have $T = \frac{1}{4}(1 + 4\pi u) : J^2 :$, so that

$$\begin{aligned} L_{-2}|h_\lambda\rangle &= \frac{1}{4}(1 + 4\pi u)(2qJ_{-2} + J_{-1}^2)|h_\lambda\rangle \\ L_{-1}^2|h_\lambda\rangle &= \frac{1}{4}(1 + 4\pi u)^2(q^2J_{-1}^2 + k(u)qJ_{-2})|h_\lambda\rangle \\ J_{-1}L_{-1}|h_\lambda\rangle &= \frac{1}{2}q(1 + 4\pi u)J_{-1}^2, \end{aligned}$$

where $k(u) = 2/(1 + 4\pi u)$. Then

$$(2L_{-2} - 2L_{-1}^2)|h_\lambda\rangle = (q_\lambda^{-1} - (1 + 4\pi u)q_\lambda)|h_\lambda\rangle.$$

Thus when $u = 0$ the state $|h_{\lambda^*}\rangle$ satisfies the ordinary level 2 null condition, corresponding to SLE_4 , while for $u \neq 0$ it satisfies the deformed relation (9), corresponding to $\text{SLE}(4, \rho)$, and its conformal weight is modified accordingly. On the other hand $|h_\lambda\rangle$ with $\lambda = \lambda^*(1 + 4\pi u)^{-1/2}$ satisfies the undeformed condition, for all u .

However, for $u \neq 0$ there is partial screening (or anti-screening) which happens in such a way that the effective current is that which would emerge from a boundary condition changing operator with the universal scaling dimension $\frac{1}{4}$.

3.2 Universal jump across level lines

This part is speculative in nature. Many of the conjectures have in fact been proved by Schramm and Sheffield[13]. So far we have shown only that the equations (5,9) hold in CFT for boundary condition changing operators in a free field theory, and that they have the same form as satisfied by the highest weight states for SLE_4 if $\lambda = \lambda^*$ and for $\text{SLE}(4, \vec{\rho})$ otherwise. We have not yet identified the curves which are described by these SLEs. A natural conjecture is that these are the level lines of the free field ϕ . For a free gaussian field theory on a triangular lattice, we can condition the field $\phi(r)$ on the existence of a curve γ which is a level line of height ϕ_0 by demanding that $\phi(r) > \phi_0$ for sites r immediately to the right of the curve and $\phi(r) < \phi_0$ for sites immediately to its left. If we impose a jump in the boundary conditions at the origin, there will be such a level line connecting the origin to infinity for each value of ϕ_0 satisfying $\phi(0-) < \phi_0 < \phi(0+)$. On

the lattice these curves will in general be different, according to how ϕ_0 is chosen.

Let us first consider the case $\lambda = \lambda^*$. Which of these curves is a suitable candidate for SLE_4 ? The answer to this appears to be that they all are the same curve in the continuum limit. Recall condition (B) for SLE, that the conditional measure on $\gamma \setminus \gamma_t$ in \mathcal{D} should be the same as the unconditioned measured in $\mathcal{D} \setminus \gamma_t$ (here we assume that γ_t is simple, so $\gamma_t = K_t$). For this to be the case, the values of the field ϕ on either side of γ_t should take the same values $\phi(0-)$ and $\phi(0+)$ as on the negative and positive real axes. This is not necessarily the case in the lattice model, however. The values of the field either side are conditioned only to be respectively less than, or greater than, ϕ_0 . Given that we have argued that for $\lambda = \lambda^*$ the state satisfies the same equation in the continuum limit as does an SLE_4 , it is natural to conjecture that this particular value gives rise, in the continuum limit, to level lines across which the jump in the field is λ^* everywhere. That is, conditioning the values on either side to take values either side of ϕ_0 , together with the existence of a jump λ^* at the boundary, has the result of enforcing a macroscopic jump λ^* all the way along γ , in the continuum limit. Schramm and Sheffield[13] have in fact proved that the level lines of the free field, defined as above on the lattice, converge to SLE_4 in the continuum limit, as long as λ takes a particular value. That the jump in the field across this curve is also λ^* has been verified in simulations by S. Sheffield[13].

If $\lambda \neq \lambda^*$, the equation satisfied by the boundary state does not correspond to simple SLE, but we have shown that it does correspond to $\text{SLE}(4, \vec{\rho})$. However we argued in the previous section this is equivalent to taking $\lambda = \lambda^*$, at the same time perturbing the action by a term $\propto \int \vec{J} \cdot \vec{J} d^2z$, and that this results in partial screening of the current-current correlations. The physics of screening is more transparent when the charges are discrete rather than continuous. For this reason let us consider a discrete gaussian model on the lattice, in which the field $\phi(r)$ at each lattice site is an integer multiple of some unit $2\pi\Lambda$. This may be enforced by adding to the action a term $\int \cos(\phi/\Lambda) d^2z$, which has bulk scaling dimension $x = 1/(2g\Lambda^2)$. This is irrelevant if $x > 2$, that is $\Lambda < \lambda^*$, in which case the continuum limit of the discrete gaussian model is given by a free field theory. On the hand if $\Lambda > \lambda^*$ the perturbation is relevant, and the theory is no longer critical. For $\Lambda = \lambda^*$ it is in fact marginally irrelevant.

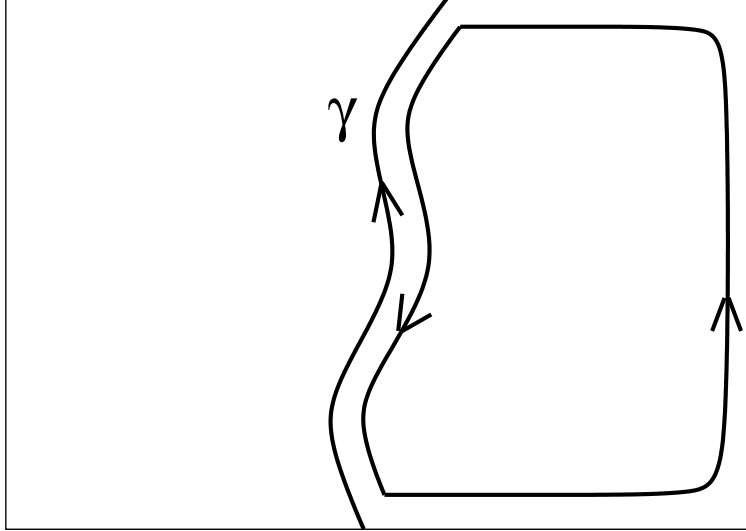


Figure 1: An energetically favoured configuration if $u > 0$, leading to screening of the current along the curve γ .

Consider therefore a discrete gaussian model with $\Lambda = \lambda^*$. On the lattice, the loops on the dual lattice carry integer currents \mathbf{I} (with the previous normalisation), and they intersect the boundary at points corresponding to boundary operators with integer charges. Such a curve with current ± 1 will correspond to a state satisfying (5) and therefore should be described by SLE_4 . Now switch on the current-current interaction. This may be modelled on the lattice by a short-range interaction $u \sum_{R,R'} f(R - R') \mathbf{I}(R) \cdot \mathbf{I}(R')$ between the currents on nearby loops. As argued in Sec. 2, this will lead to partial screening if $uf > 0$: a current \mathbf{I} will attract those parts of nearby loops with currents anti-parallel to \mathbf{I} and repel those parts with parallel currents. The resultant effective current along the curve will be reduced by a factor $(1 + 4\pi u)^{-1/2}$. Similarly, there will be anti-screening if $u < 0$. Since the currents are discrete, this can of course only happen in an average sense. In Fig. 1 we illustrate a loop configuration which would contribute to the screening phenomenon. Note that in this case there are no sources for \mathcal{J} . If there were, they could also contribute to the screening. *Acknowledgments.* I would like to thank Scott Sheffield for explaining to me some of the results of Ref. [13] before publication, and Roland Friedrich for discussions. This work was carried out while the author was a joint member of the Schools of

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References

- [1] O. Schramm, Israel J. Math. **118**, 221, 2000.
- [2] G. Lawler, O. Schramm and W. Werner, Acta Mathematica **187** 237, 2001 (math.PR/9911084); *ibid.* **187** 275, 2003 (math.PR/0003156); Ann. Henri Poincaré **38** 109, 2002 (math.PR/0005294).
- [3] S. Rohde and O. Schramm, Ann. Math., to appear (math.PR/0106036).
- [4] For reviews, see W. Werner, *Random planar curves and Schramm-Loewner evolutions*, to appear (Springer Lecture Notes) (math.PR/0303354); G. Lawler, *Conformally Invariant Processes in the Plane*, in preparation, <http://www.math.cornell.edu/~lawler/book.ps>; W. Kager and B. Nienhuis, J. Stat. Phys. **115**, 1149 2004 (math-ph/0312056); J. Cardy, *SLE for theoretical physicists*, in preparation.
- [5] A.A. Belavin, A.M. Polyakov and A.B. Zamolodchikov, Nucl. Phys. B **241**, 333, 1984.
- [6] R. Friedrich, math-ph/0410029; R. Bauer and R. Friedrich, math.PR/0408157.
- [7] J. Cardy, J. Phys. A **36**, L379, 2003 (erratum J. Phys. A **36**, 12343, 2003); Phys. Lett. B **582**, 121, 2004.
- [8] G. Lawler, O. Schramm and W. Werner, J. Amer. Math. Soc. **16**(4), 917, 2003 (math.PR/0209343).
- [9] J. Dubédat, Ann. Probab., to appear (math.PR/0303128).
- [10] W. Werner, Ann. Fac. Sci. Toulouse, to appear (math.PR/0302115).
- [11] J. Dubédat, math.PR/0405074.
- [12] G. Watts, J. Phys. A **29**, L363, 1996 (cond-mat/9603167).

- [13] O. Schramm and S. Sheffield, in preparation; S. Sheffield, math.PR/0312099 and talk presented at ‘Conformal Invariance and Random Spatial Processes’, Edinburgh, July 2003.
- [14] M. Bauer and D. Bernard, Comm. Math. Phys. **239**, 493, 2003 (hep-th/0210015); Phys. Lett. B **543**, 135, 2002; Phys. Lett. B **557**, 309, 2003 (hep-th/0301064); Ann. Henri Poincaré **5**, 289, 2004 (math-ph/0305061).
- [15] J. Cardy, Nucl. Phys. B **324**, 581, 1989.
- [16] B. Nienhuis, J. Stat. Phys. **34**, 731, 1983.
- [17] Vl. Dotsenko and V. Fateev, Nucl. Phys. **B240**, 312, 1984; *ibid.* **B251**, 691, 1985.
- [18] J. Kondev, Phys. Rev. Lett. **78**, 4320, 1997.